12.2 Example (1)

First order wave equation (1)

$$\begin{cases} u_t + a\,u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$$\boxed{u(x,t) = \phi(x-at)}$$

Any "a" and any I.C. $\phi(x)$. differentiable w.r.t. x and t

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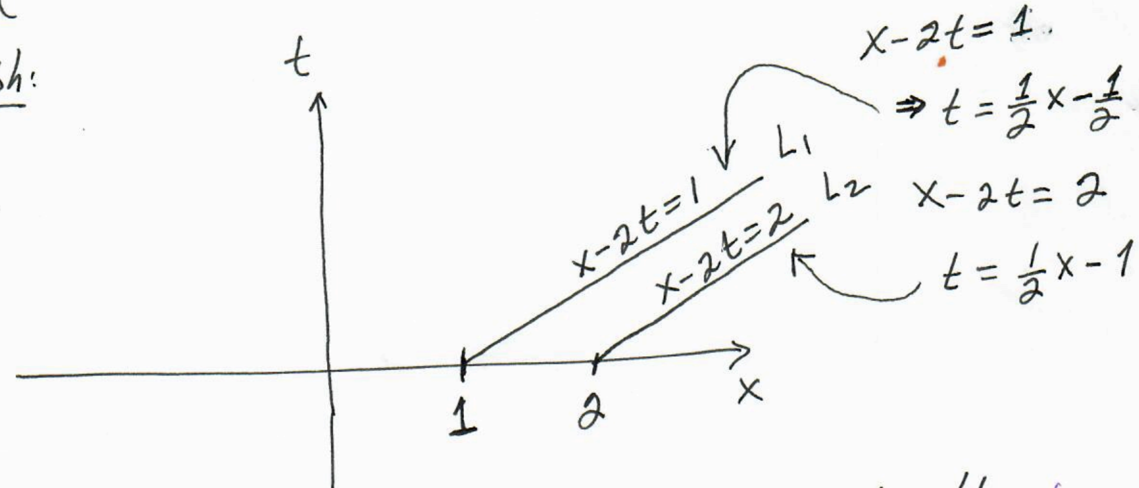
Particular case:

$$\begin{cases} u_t + 2u_x = 0 \\ u(x,0) = 4x \end{cases} \Rightarrow u(x,t) = 4(x-2t).$$

Graph:



on L_1: $\quad u(x,t) = 4(x-2t) = 4*1 = 4$, for all $(x,t)\in L_1$

on L_2: $\quad u(x,t) = 4(x-2t) = 4*2 = 8$, for all $(x,t)\in L_2$

Discuss wave behavior.

We can show now that

$$u(x,t) = \phi(x-at)$$

is indeed a Soln. of (1)-(2) if $\phi(x)$ has first derivatives. It might be points (x,t) where the Soln of (1) does not exist (jump discontinuity for example).

In fact,

$$u_t = \phi'(x-at)\,\frac{d}{dt}(x-at) = \phi'(x-at)(-a)$$

$$u_x = \phi'(x-at)\,\frac{d}{dx}(x-at) = \phi'(x-at)\,(1)$$

$$\Rightarrow \quad u_t + a\,u_x = -a\,\phi'(x-at) + a\,\phi'(x-at) = 0. \checkmark$$

Also I.C. is Satisfied since

$$u(x,0) = \phi(x-a0) = \phi(x). \checkmark$$

<u>Application:</u> Solve the IVP.

$$\begin{cases} u_t + 2u_x = 0, & -\infty < x < \infty \\ u(x,0) = \begin{cases} 4x, & 0 < x < 1 \\ 0, & x < 0 \text{ or } x > 1 \end{cases} \end{cases}$$

IVP

$$\begin{cases} u_t + 2u_x = 0 \\ u(x,0) = \begin{cases} 4x, & 0<x<1 \\ 0, & \text{otherwise.} \end{cases} \end{cases}$$

Soln:

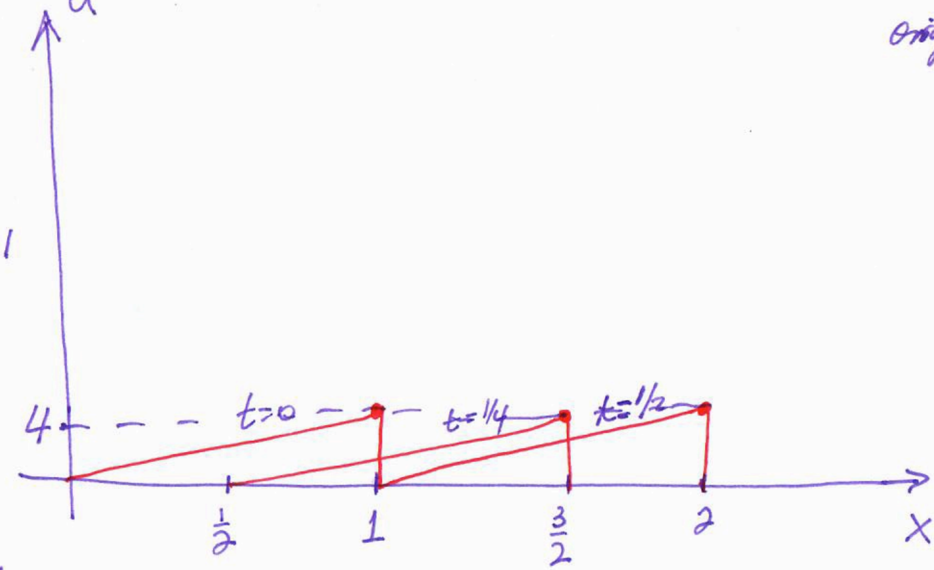
$$u(x,t) = \begin{cases} 4(x-2t), & 0<x-2t<1 \\ 0, & \text{otherwise} \end{cases}$$

Explain using Floor of room. Select a corner as the origin of coords.

at $t=1/2$, $0<x-2(\frac{1}{2})<1$

$1<x<2$

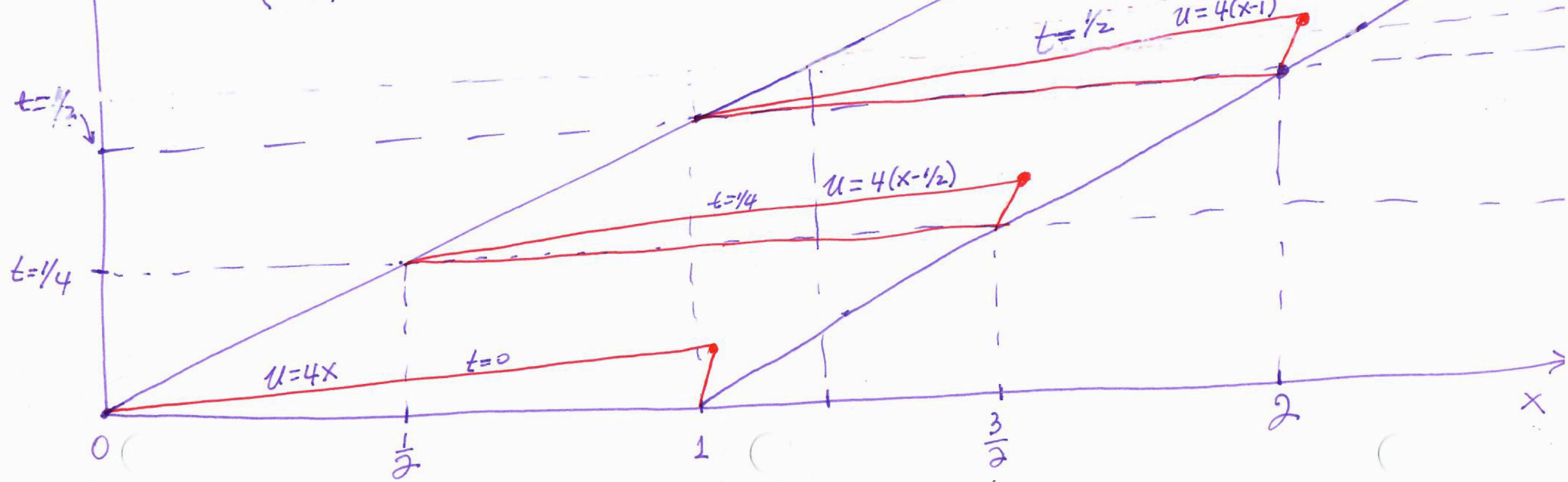
$$u(x,t) = \begin{cases} 4(x-1), & 1<x<2 \\ 0, & \text{otherwise} \end{cases}$$



at $t=1/4$, $0<x-2(\frac{1}{4})<1$

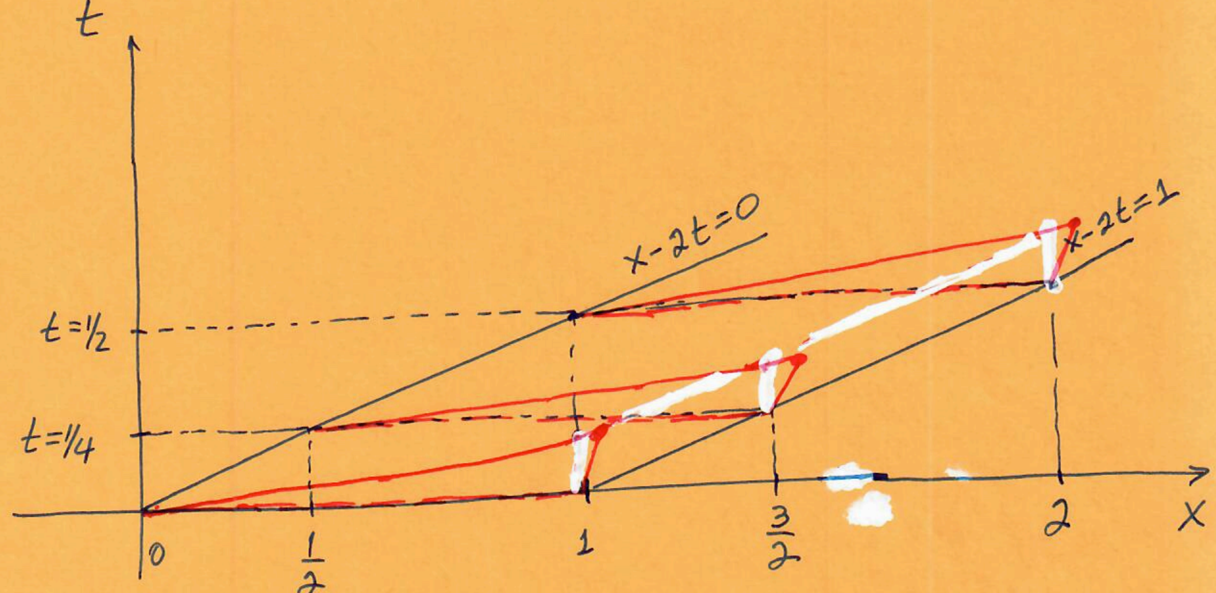
$\frac{1}{2}<x<\frac{3}{2}$

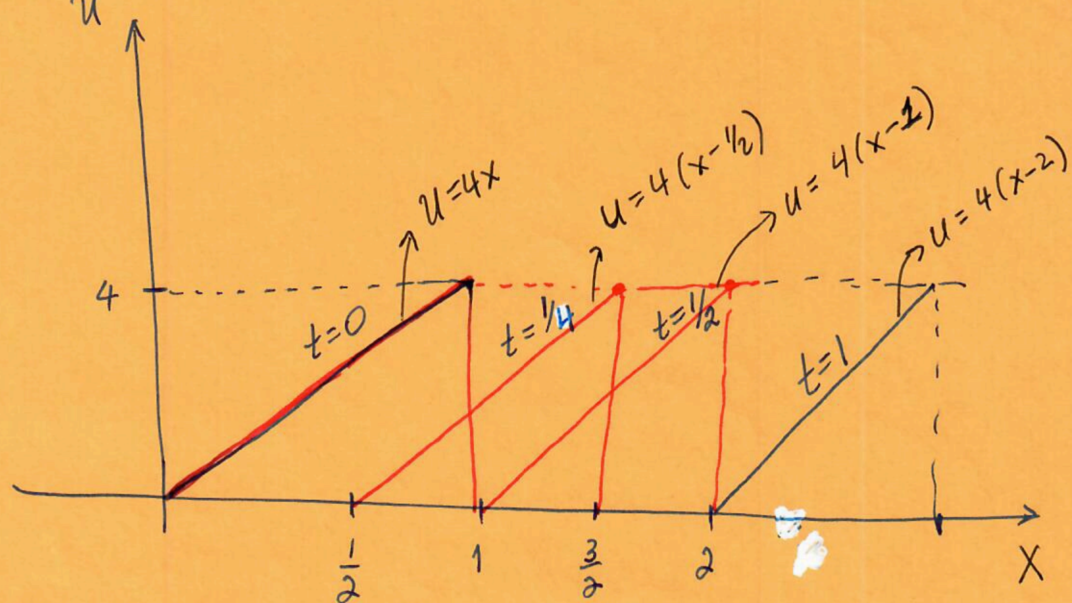
$$\Rightarrow u(x,1/4) = \begin{cases} 4(x-\frac{1}{2}), & \frac{1}{2}<x<\frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$



From the theory above,

$$u(x,t) = \phi(x-2t) = \begin{cases} 4(x-2t), & 0 < x-2t < 1 \\ 0, & x-2t < 0 \text{ or } x-2t > 1 \end{cases}$$





Rmk: We learn from this example that our Solution may be discontinuous along some Characteristics, then, equation (1) or the I.C. (2) may not be Satisfied.

Method of Characteristic. First order PDE.

Consider the advection equation:

$$\begin{cases} u_t + a u_x = 0, & a>0, \quad -\infty < x < \infty, \ t>0 \quad (1) \\ u(x,0) = \phi(x), & -\infty < x < \infty, \quad \phi \text{ is differentiable or even piecewise smooth} \quad (2) \end{cases}$$

Thm.- The Soln. of Initial Value problem (1)-(2) is given by

$$u(x,t) = \phi(x-at), \quad -\infty<x<\infty, \quad t>0.$$

Proof.- Assume ① $u(x,t)$ is a soln of (1)-(2)

② The point (x^*, t^*) is an arbitrary point in the semiplane. $D=\{(x,t): -\infty<x<\infty, t\geq 0\}$.

③ Consider a curve $\mathcal{C}: x = x(t)$ passing through (x^*, t^*).

then, along the curve $\mathcal{C}$,

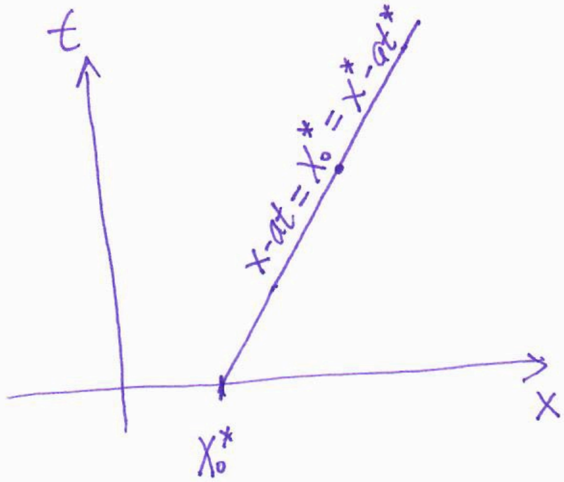
$$\boxed{u(x,t) = u(x(t),t) = \hat{u}(t)} \qquad (2.2)$$

and $$\frac{d\hat{u}}{dt}(t) \overset{\text{chain rule}}{=} \frac{\partial u}{\partial x}(x(t),t)\,\frac{dx}{dt}(t) + \frac{\partial u}{\partial t}(x(t),t) \qquad (3)$$

If we require $\mathcal{C}$ to satisfy $\boxed{\frac{dx}{dt}(t) = a} \overset{(3.1)}{\Rightarrow} \boxed{x(t) = at + x_0^*}$

In other words, $\mathcal{C}$ is the line: $\boxed{x - at = x_0^*}$ (3.2)

Since $(x^*, t^*) \in \mathcal{C} \Rightarrow \boxed{x^* - at^* = x_0^*}$ (3.3)



Combining (3) and (3.1), we are led to

$$\frac{d\hat{u}}{dt}(t) = \frac{\partial u}{\partial x}(x(t),t)\,\frac{dx}{dt}(t) + \frac{\partial u}{\partial t}(x(t),t) =$$

$$= a\,\frac{\partial u}{\partial x}(x(t),t) + \frac{\partial u}{\partial t}(x(t),t) \overset{\text{from (1)}}{=} 0$$

Thus, $\boxed{\hat{u}(t) \equiv K}$, $\qquad$ (4.1)

Using (2.2), we arrive to

$\boxed{u(x,t) = \hat{u}(t) = K,}$ for all $(x,t) \in \mathcal{C}$. (4.11)

It means the solution $u(x,t)$ is constant along $\mathcal{C}$.

Since $x - at = x_0^*$ on $\mathcal{C} \Rightarrow$ at $t=0$, $x = x_0^*$

Then, $(x_0^*, 0) \in \mathcal{C}$ and $\boxed{u(x_0^*, 0) = K}$ $\qquad$ (4.2)

But also, using the I.C. (2)

$$\boxed{u(x_0^*, 0) = \phi(x_0^*)} \qquad (4.3)$$

From (4.3), (4.2), and (3.2)

$$K = \phi(x_0^*) = \phi(x-at), \quad \text{for all } (x,t)\in \mathcal{C}.$$

Subst. into (4.1) leads to

$$\boxed{u(x,t) = \phi(x-at),} \quad \text{for all } (x,t)\in \mathcal{C}$$

In particular,

$$\boxed{u(x^*,t^*) = \phi(x^*-at^*),} \quad \text{for our arbitrary } (x^*,t^*)\in \underset{\substack{\uparrow \\ \text{Semiplane.}}}{D}$$

Therefore,

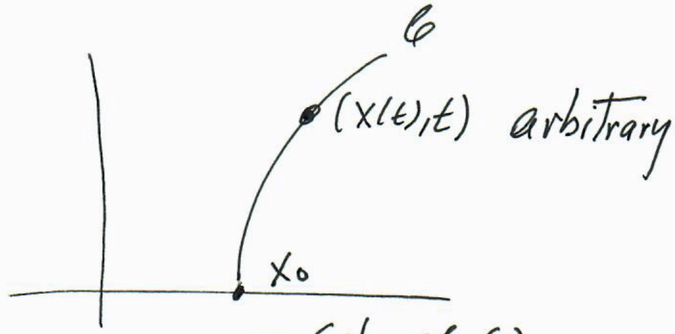
$$\boxed{u(x,t) = \phi(x-at),} \quad \text{for all } (x,t)\in D.$$

Def. The Line $L:\ x-at = x_0^*$ is called a characteristic line for the IVP (1)-(2).

Rmk: For any point (x,t) in the Semiplane D, there is a characteristic line passing through it.

My problem (in class)

$$\begin{cases} u_t + 4u_x = 1+x & (1) \\ u(x,0) = e^{3x} & (2) \end{cases}$$



$$\hat{u}(t) = \underbrace{u}_{\text{soln. of (1)}}(x(t),t) \text{ on } \mathcal{C}$$

$$\frac{d\hat{u}}{dt} = u_t + u_x \frac{dx}{dt}$$

If $\frac{dx}{dt} = 4 \Rightarrow \frac{d\hat{u}}{dt} = u_t + u_x(4) = 1 + x(t)$ on $\mathcal{C}$.

Then,

1) $x(t) = 4t + d$

If $t = 0 \Rightarrow x_0 = 4(0) + d \Rightarrow \boxed{d = x_0}$

and $\underline{\text{charact.}}$ $x(t) = 4t + x_0$

or $\boxed{x - 4t = x_0}$

2) $\frac{d\hat{u}}{dt}(t) = 1 + x(t) = 1 + 4t + x_0$

$\Rightarrow \hat{u}(t) = t + 2t^2 + x_0 t + f$

or $\quad u(x(t),t) = (1+x_0)t + 2t^2 + f$

Using I.C.: $\quad e^{3x_0} = u(x(0),0) = f$

Since $\quad x_0 = x - 4t \Rightarrow$

$$u(x(t),t) = (1 + x - 4t)t + 2t^2 + e^{3(x-4t)}$$

Since $(x(t),t)$ is arbitrary,

$$\boxed{u(x,t) = 1 + xt - 2t^2 + e^{3(x-4t)}}$$

<u>Verification</u>: $\quad u_t = 1 + x - 4t - 12e^{3(x-4t)}$

$$u_x = t + 3e^{3(x-4t)}$$

$$\Rightarrow u_t + 4u_x = 1 + x - \cancel{4t} + \cancel{4t} - \cancel{12e^{3(x-4t)}} + \cancel{(4)3}e^{3(x-4t)} = 1 + x \checkmark$$

$$u(x,0) = e^{3x} \checkmark$$